Geometric aspects of quantum computing

Maris Ozols

University of Waterloo
Department of C&O

December 10, 2007

Qubit state

Qubit

$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

Qubit state

Qubit

$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

Parametrization

We can find θ ($0 \leq \theta \leq \pi$) and φ ($0 \leq \varphi < 2\pi$), such that

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}.$$

Qubit state

Qubit

$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

Parametrization

We can find θ $(0 \leq \theta \leq \pi)$ and φ $(0 \leq \varphi < 2\pi)$, such that

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}.$$

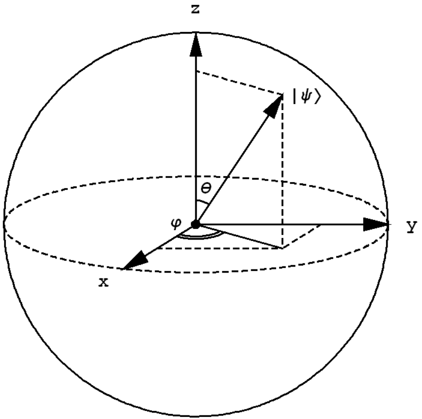
Density matrix

The corresponding density matrix is:

$$\rho = |\psi\rangle\,\langle\psi| = \frac{1}{2}\begin{pmatrix} 1+\cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & 1-\cos\theta \end{pmatrix}.$$

Bloch sphere

Bijection between S^2 and $\mathbb{CP}^1$



$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$\Updownarrow$$

$$\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$$

$0 \le \theta \le \pi$ and $0 \le \varphi < 2\pi$

Pauli matrices

Density matrix of a qubit

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right), \quad \vec{r} = (x, y, z), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

Pauli matrices

Density matrix of a qubit

$$\rho = \frac{1}{2}\left(I + \vec{r} \cdot \vec{\sigma}\right), \quad \vec{r} = (x, y, z), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Pauli matrices

Density matrix of a qubit

$$\rho = \frac{1}{2}\left(I + \vec{r} \cdot \vec{\sigma}\right), \quad \vec{r} = (x, y, z), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

Pauli matrices

$$I = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \quad \sigma_x = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \quad \sigma_y = \left(\begin{smallmatrix} 0 & -i \\ i & 0 \end{smallmatrix}\right), \quad \sigma_z = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right).$$

Inner product

$$|\langle\psi_1|\psi_2\rangle|^2$$

Pauli matrices

Density matrix of a qubit

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right), \quad \vec{r} = (x, y, z), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Inner product

$$|\langle\psi_1|\psi_2\rangle|^2 = \mathrm{Tr}(\rho_1\rho_2)$$

Pauli matrices

Density matrix of a qubit

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right), \quad \vec{r} = (x, y, z), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Inner product

$$|\langle\psi_1|\psi_2\rangle|^2 = \mathrm{Tr}(\rho_1\rho_2) = \frac{1}{2}(1 + \vec{r}_1\cdot\vec{r}_2).$$

Pauli matrices

Density matrix of a qubit

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right), \quad \vec{r} = (x, y, z), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Inner product

$$\underbrace{|\langle\psi_1|\psi_2\rangle|^2}_{0} = \mathrm{Tr}(\rho_1\rho_2) = \frac{1}{2}(1 + \vec{r}_1\cdot\vec{r}_2).$$

Pauli matrices

Density matrix of a qubit

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right), \quad \vec{r} = (x, y, z), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Inner product

$$\underbrace{\left|\langle\psi_1|\psi_2\rangle\right|^2}_{0} = \mathrm{Tr}(\rho_1\rho_2) = \frac{1}{2}(1 + \underbrace{\vec{r}_1\cdot\vec{r}_2}).$$

Pauli matrices

Density matrix of a qubit

$$\rho = \frac{1}{2}\left(I + \vec{r}\cdot\vec{\sigma}\right), \quad \vec{r} = (x, y, z), \quad \vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z).$$

Pauli matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Inner product

$$\underbrace{|\langle\psi_1|\psi_2\rangle|^2}_{0} = \mathrm{Tr}(\rho_1\rho_2) = \frac{1}{2}(1 + \underbrace{\vec{r}_1\cdot\vec{r}_2}_{-1}).$$

General 2×2 unitary

Rotation around z-axis

Consider the action of $U = |0\rangle\langle 0| + e^{i\varphi}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$:

General 2×2 unitary

Rotation around z-axis

Consider the action of $U = |0\rangle\langle 0| + e^{i\varphi}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$:

$$U|0\rangle = |0\rangle , \quad U|1\rangle = e^{i\varphi}|1\rangle .$$

General 2×2 unitary

Rotation around z-axis

Consider the action of $U = |0\rangle\langle 0| + e^{i\varphi}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$:

$$U|0\rangle = |0\rangle, \quad U|1\rangle = e^{i\varphi}|1\rangle.$$

Now if we act on $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, we get:

$$U\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{i\varphi}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix}.$$

General 2×2 unitary

Rotation around z-axis

Consider the action of $U = |0\rangle\langle 0| + e^{i\varphi}|1\rangle\langle 1| = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$:

$$U|0\rangle = |0\rangle\,, \quad U|1\rangle = e^{i\varphi}|1\rangle\,.$$

Now if we act on $\frac{|0\rangle+|1\rangle}{\sqrt{2}}$, we get:

$$U\frac{|0\rangle + |1\rangle}{\sqrt{2}} = \frac{|0\rangle + e^{i\varphi}|1\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ e^{i\varphi} \end{pmatrix}.$$

General rotation

Rotation around $\vec{r}$ by angle φ:

$$U(\vec{r}, \varphi) = \rho(\vec{r}) + e^{i\varphi}\rho(-\vec{r}).$$

Some curiosities

Some curiosities

Quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Some curiosities

Quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Bijection

$$(1, i, j, k) \Longleftrightarrow (I, i\sigma_z, i\sigma_y, i\sigma_x).$$

Some curiosities

Quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Bijection

$$(1, i, j, k) \Longleftrightarrow (I, i\sigma_z, i\sigma_y, i\sigma_x).$$

Some curiosities

Quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Bijection

$$(1, i, j, k) \Longleftrightarrow (I, i\sigma_z, i\sigma_y, i\sigma_x).$$

Finite field of order 4

$$\mathbb{F}_4 = (\left\{0, 1, \omega, \omega^2\right\}, +, *), \quad x \equiv -x, \quad \omega^2 \equiv \omega + 1.$$

Some curiosities

Quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Bijection

$$(1, i, j, k) \Longleftrightarrow (I, i\sigma_z, i\sigma_y, i\sigma_x).$$

Finite field of order 4

$$\mathbb{F}_4 = (\left\{0, 1, \omega, \omega^2\right\}, +, *), \quad x \equiv -x, \quad \omega^2 \equiv \omega + 1.$$

Bijection

$$(0, 1, \omega, \omega^2) \Longleftrightarrow (I, \sigma_x, \sigma_z, \sigma_y).$$

Some curiosities

Quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Bijection

$$(1, i, j, k) \iff (I, i\sigma_z, i\sigma_y, i\sigma_x).$$

Finite field of order 4

$$\mathbb{F}_4 = (\left\{0, 1, \omega, \omega^2\right\}, +, *), \quad x \equiv -x, \quad \omega^2 \equiv \omega + 1.$$

Bijection (up to phase)

$$(0, 1, \omega, \omega^2) \iff (I, \sigma_x, \sigma_z, \sigma_y).$$

Some curiosities

Quaternions

$$i^2 = j^2 = k^2 = ijk = -1.$$

Bijection

$$(1, i, j, k) \Longleftrightarrow (I, i\sigma_z, i\sigma_y, i\sigma_x).$$

Finite field of order 4

$$\mathbb{F}_4 = (\left\{0, 1, \omega, \omega^2\right\}, +, *), \quad x \equiv -x, \quad \omega^2 \equiv \omega + 1.$$

Bijection (up to phase)

$$(0, 1, \omega, \omega^2) \Longleftrightarrow (I, \sigma_x, \sigma_z, \sigma_y).$$

More curiosities

More curiosities

Clifford group

The *Clifford group* of a qubit is

$$C = \{U|\sigma \in P \Rightarrow U\sigma U^\dagger \in P\},$$

where $P = \{\pm I, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z\}$ – the set of Pauli matrices.

More curiosities

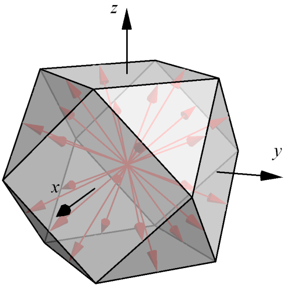
Clifford group

The *Clifford group* of a qubit is

$$C = \{U | \sigma \in P \Rightarrow U \sigma U^{\dagger} \in P\},$$

where $P = \{\pm I, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z\}$ – the set of Pauli matrices.

Cuboctahedron



More curiosities

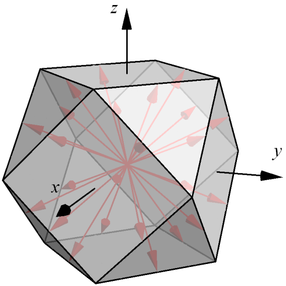
Clifford group

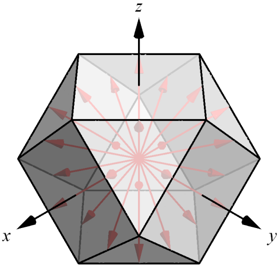
The *Clifford group* of a qubit is

$$C = \{U | \sigma \in P \Rightarrow U\sigma U^{\dagger} \in P\},$$

where $P = \{\pm I, \pm\sigma_x, \pm\sigma_y, \pm\sigma_z\}$ – the set of Pauli matrices.

Cuboctahedron





Counting dimensions

Counting dimensions

Density matrix

Counting dimensions

Density matrix

1. hermitian: $\rho^\dagger = \rho$,

Counting dimensions

Density matrix

1. hermitian: $\rho^\dagger = \rho$,
2. unit trace: $\operatorname{Tr} \rho = 1$,

Counting dimensions

Density matrix

1. hermitian: $\rho^\dagger = \rho$,
2. unit trace: $\mathrm{Tr}\,\rho = 1$,
3. positive semi-definite: $\rho \geq 0$.

Counting dimensions

Density matrix

1. hermitian: $\rho^\dagger = \rho$,
2. unit trace: $\operatorname{Tr} \rho = 1$,
3. positive semi-definite: $\rho \geq 0$.

Degrees of freedom for ρ

For an $n \times n$ density matrix there are $n^2 - 1$ degrees of freedom.

Counting dimensions

Density matrix

1. hermitian: $\rho^\dagger = \rho$,
2. unit trace: $\mathrm{Tr}\,\rho = 1$,
3. positive semi-definite: $\rho \geq 0$.

Degrees of freedom for ρ

For an $n \times n$ density matrix there are $n^2 - 1$ degrees of freedom.

Degrees of freedom for $|\psi\rangle$

A pure quantum state $|\psi\rangle \in \mathbb{C}^n$ has $2(n-1)$ degrees of freedom.

Qutrit

Qutrit state

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle + \gamma |2\rangle .$$

Qutrit

Qutrit state

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle + \gamma\,|2\rangle\,.$$

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Qutrit

Qutrit state

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle + \gamma\,|2\rangle\,.$$

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$
$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Density matrix

$$\rho = |\psi\rangle\,\langle\psi| = \frac{1}{3}(I + \sqrt{3}\,\vec{r}\cdot\vec{\lambda}), \quad \vec{r} \in \mathbb{R}^8.$$

Qutrit

Qutrit state

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle + \gamma\,|2\rangle\,.$$

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Density matrix

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{3}(I + \sqrt{3}\,\vec{r}\cdot\vec{\lambda}), \quad \vec{r} \in \mathbb{R}^8.$$

Qutrit

Qutrit state

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle + \gamma\,|2\rangle\,.$$

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Density matrix

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{3}(I + \sqrt{3}\,\vec{r}\cdot\vec{\lambda}), \quad \vec{r} \in \mathbb{R}^8.$$

Qutrit

Qutrit state

$$|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle + \gamma\,|2\rangle\,.$$

Gell-Mann matrices

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \qquad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \qquad \lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Density matrix

$$\rho = |\psi\rangle\langle\psi| = \frac{1}{3}(I + \sqrt{3}\,\vec{r}\cdot\vec{\lambda}), \quad \vec{r} \in \mathbb{R}^8.$$

$\mathrm{SU}(n)$ generators

Generalized Pauli matrices

Let $P_{jk} = |j\rangle \langle k|$.

SU(n) generators

Generalized Pauli matrices

Let $P_{jk} = |j\rangle \langle k|$. Define

$$\begin{cases} X_{jk} = P_{kj} + P_{jk}, \\ Y_{jk} = i(P_{kj} - P_{jk}), \\ Z_j = \sqrt{\frac{2}{j(j+1)}} \left(P_{11} + P_{22} + \cdots + P_{jj} - j \cdot P_{j+1,j+1}\right), \end{cases}$$

where $1 \leq j < k \leq n$.

SU(n) generators

Generalized Pauli matrices

Let $P_{jk} = |j\rangle\langle k|$. Define

$$\begin{cases} X_{jk} = P_{kj} + P_{jk}, \\ Y_{jk} = i(P_{kj} - P_{jk}), \\ Z_j = \sqrt{\frac{2}{j(j+1)}}\left(P_{11} + P_{22} + \cdots + P_{jj} - j \cdot P_{j+1,j+1}\right), \end{cases}$$

where $1 \leq j < k \leq n$. Then

$$\{\lambda_i\} = \{X_{jk}\} \cup \{Y_{jk}\} \cup \{Z_j\}$$

is the set of *generalized Pauli matrices*. Note that $|\{\lambda_i\}| = n^2 - 1$.

State of an n-level quantum system

General density matrix

$$\rho = \frac{1}{n}\left(I + \sqrt{\frac{n(n-1)}{2}}\,\vec{r}\cdot\vec{\lambda}\right), \quad \vec{r} \in \mathbb{R}^{n^2-1}.$$

State of an n-level quantum system

General density matrix

$$\rho = \frac{1}{n}\left(I + \sqrt{\frac{n(n-1)}{2}}\,\vec{r}\cdot\vec{\lambda}\right), \quad \vec{r} \in \mathbb{R}^{n^2-1}.$$

Inner product

$$|\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{n}\Big(1 + (n-1)\,\vec{r}_1\cdot\vec{r}_2\Big)$$

State of an n-level quantum system

General density matrix

$$\rho = \frac{1}{n}\left(I + \sqrt{\frac{n(n-1)}{2}}\,\vec{r}\cdot\vec{\lambda}\right), \quad \vec{r} \in \mathbb{R}^{n^2-1}.$$

Inner product

$$0 \leq \, |\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{n}\Big(1 + (n-1)\,\vec{r}_1\cdot\vec{r}_2\Big) \leq 1,$$

State of an n-level quantum system

General density matrix

$$\rho = \frac{1}{n}\left(I + \sqrt{\frac{n(n-1)}{2}}\,\vec{r}\cdot\vec{\lambda}\right), \quad \vec{r} \in \mathbb{R}^{n^2-1}.$$

Inner product

$$0 \leq \left|\langle\psi_1|\psi_2\rangle\right|^2 = \frac{1}{n}\Big(1 + (n-1)\,\vec{r}_1\cdot\vec{r}_2\Big) \leq 1,$$

$$-\frac{1}{n-1} \leq \vec{r}_1\cdot\vec{r}_2 \leq 1.$$

State of an n-level quantum system

General density matrix

$$\rho = \frac{1}{n}\left(I + \sqrt{\frac{n(n-1)}{2}}\,\vec{r}\cdot\vec{\lambda}\right), \quad \vec{r} \in \mathbb{R}^{n^2-1}.$$

Inner product

$$0 \leq |\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{n}\Big(1 + (n-1)\,\vec{r}_1\cdot\vec{r}_2\Big) \leq 1,$$

$$-\frac{1}{n-1} \leq \vec{r}_1\cdot\vec{r}_2 \leq 1.$$

Orthogonal states

$$\vec{r}_1\cdot\vec{r}_2 = \cos\theta = -\frac{1}{n-1}.$$

State of an n-level quantum system

General density matrix

$$\rho = \frac{1}{n}\left(I + \sqrt{\frac{n(n-1)}{2}}\,\vec{r}\cdot\vec{\lambda}\right), \quad \vec{r} \in \mathbb{R}^{n^2-1}.$$

Inner product

$$0 \leq |\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{n}\Big(1 + (n-1)\,\vec{r}_1\cdot\vec{r}_2\Big) \leq 1,$$

$$-\frac{1}{n-1} \leq \vec{r}_1\cdot\vec{r}_2 \leq 1.$$

Orthogonal states

$$\vec{r}_1\cdot\vec{r}_2 = \cos\theta = -\frac{1}{n-1}.$$

Unit simplex

Unit simplex

Geometry

Height H_n, circumradius R_n, inradius r_n:

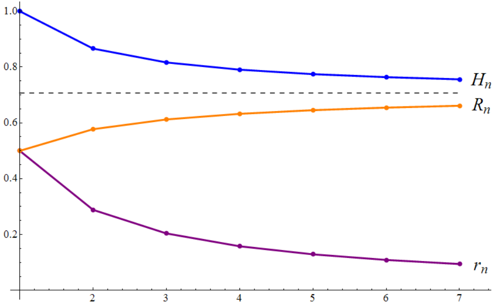
$$H_n = \sqrt{\tfrac{n+1}{2n}}, \quad R_n = \sqrt{\tfrac{n}{2(n+1)}}, \quad r_n = \frac{1}{\sqrt{2n(n+1)}}.$$

Unit simplex

Geometry

Height H_n, circumradius R_n, inradius r_n:

$$H_n = \sqrt{\frac{n+1}{2n}}, \quad R_n = \sqrt{\frac{n}{2(n+1)}}, \quad r_n = \frac{1}{\sqrt{2n(n+1)}}.$$

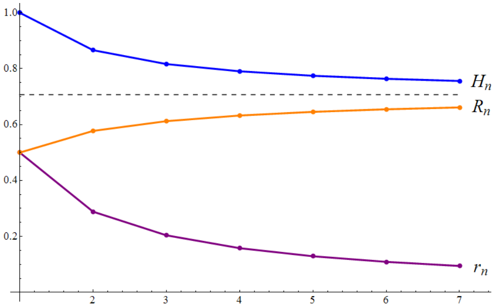


Unit simplex

Geometry

Height H_n, circumradius R_n, inradius r_n:

$$H_n = \sqrt{\frac{n+1}{2n}}, \quad R_n = \sqrt{\frac{n}{2(n+1)}}, \quad r_n = \frac{1}{\sqrt{2n(n+1)}}.$$



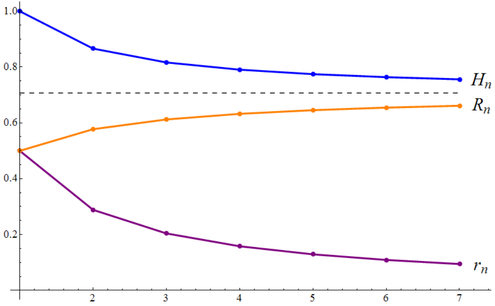
$$H_n = R_n + r_n$$

Unit simplex

Geometry

Height H_n, circumradius R_n, inradius r_n:

$$H_n = \sqrt{\frac{n+1}{2n}}, \quad R_n = \sqrt{\frac{n}{2(n+1)}}, \quad r_n = \frac{1}{\sqrt{2n(n+1)}}.$$



$$H_n = R_n + r_n$$

$$\frac{R_n}{r_n} = n$$

Unit vectors

Unit vectors

Simplex of unit vectors

Unit vectors $\{\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n\}$ in $\mathbb{R}^n$ form a *regular simplex* if

$$\vec{v}_i \cdot \vec{v}_j = -\frac{1}{n} \quad (i \neq j).$$

Unit vectors

Simplex of unit vectors

Unit vectors $\{\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n\}$ in $\mathbb{R}^n$ form a *regular simplex* if

$$\vec{v}_i \cdot \vec{v}_j = -\frac{1}{n} \quad (i \neq j).$$

Orthonormal basis in $\mathbb{C}^n$

An orthonormal basis $\mathcal{B} = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\}$ of $\mathbb{C}^n$ satisfy:

$$|\langle\psi_i|\psi_j\rangle|^2 = 0, \quad \vec{v}_i \cdot \vec{v}_j = -\frac{1}{n-1} \quad (i \neq j).$$

Unit vectors

Simplex of unit vectors

Unit vectors $\{\vec{v}_0, \vec{v}_1, \ldots, \vec{v}_n\}$ in $\mathbb{R}^n$ form a *regular simplex* if

$$\vec{v}_i \cdot \vec{v}_j = -\frac{1}{n} \quad (i \neq j).$$

Orthonormal basis in $\mathbb{C}^n$

An orthonormal basis $\mathcal{B} = \{|\psi_1\rangle, |\psi_2\rangle, \ldots, |\psi_n\rangle\}$ of $\mathbb{C}^n$ satisfy:

$$|\langle\psi_i|\psi_j\rangle|^2 = 0, \quad \vec{v}_i \cdot \vec{v}_j = -\frac{1}{n-1} \quad (i \neq j).$$

It is a regular simplex in $n-1$ dimensional subspace of $\mathbb{R}^{n^2-1}$!

Unit vectors

Simplex of unit vectors

Unit vectors $\{\vec{v}_0, \vec{v}_1, \dots, \vec{v}_n\}$ in $\mathbb{R}^n$ form a *regular simplex* if

$$\vec{v}_i \cdot \vec{v}_j = -\frac{1}{n} \quad (i \neq j).$$

Orthonormal basis in $\mathbb{C}^n$

An orthonormal basis $\mathcal{B} = \{|\psi_1\rangle, |\psi_2\rangle, \dots, |\psi_n\rangle\}$ of $\mathbb{C}^n$ satisfy:

$$|\langle\psi_i|\psi_j\rangle|^2 = 0, \quad \vec{v}_i \cdot \vec{v}_j = -\frac{1}{n-1} \quad (i \neq j).$$

It is a regular simplex in $n-1$ dimensional subspace of $\mathbb{R}^{n^2-1}$!

MUBs

MUBs

Mutually Unbiased Bases

A set of $n+1$ orthonormal bases $\mathcal{B}_i = \left\{ \left|\psi_1^i\right\rangle, \left|\psi_2^i\right\rangle, \ldots, \left|\psi_n^i\right\rangle \right\}$ of $\mathbb{C}^n$, such that for any $|\psi_i\rangle \in \mathcal{B}_i$ and $|\psi_j\rangle \in \mathcal{B}_j$

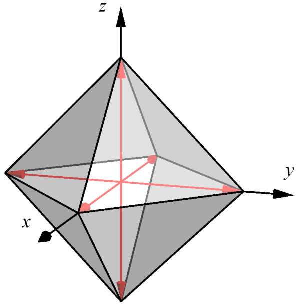
$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n}, \quad \vec{v}_i \cdot \vec{v}_j = 0 \quad (i \neq j).$$

MUBs

Mutually Unbiased Bases

A set of $n+1$ orthonormal bases $\mathcal{B}_i = \{|\psi_1^i\rangle, |\psi_2^i\rangle, \ldots, |\psi_n^i\rangle\}$ of $\mathbb{C}^n$, such that for any $|\psi_i\rangle \in \mathcal{B}_i$ and $|\psi_j\rangle \in \mathcal{B}_j$

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n}, \quad \vec{v}_i \cdot \vec{v}_j = 0 \quad (i \neq j).$$

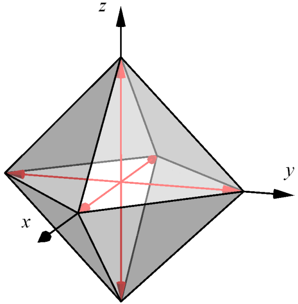


MUBs

Mutually Unbiased Bases

A set of $n+1$ orthonormal bases $\mathcal{B}_i = \left\{\left|\psi_1^i\right\rangle, \left|\psi_2^i\right\rangle, \ldots, \left|\psi_n^i\right\rangle\right\}$ of $\mathbb{C}^n$, such that for any $|\psi_i\rangle \in \mathcal{B}_i$ and $|\psi_j\rangle \in \mathcal{B}_j$

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n}, \quad \vec{v}_i \cdot \vec{v}_j = 0 \quad (i \neq j).$$



$$\mathcal{B}_z = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

$$\mathcal{B}_x = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\},$$

$$\mathcal{B}_y = \left\{ \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}.$$

Duality

Duality

Unbiased $\Longrightarrow$ Orthogonal

If $|\psi_i\rangle$ and $|\psi_j\rangle$ are from different bases, then

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n}, \quad \vec{v}_i \cdot \vec{v}_j = 0 \quad (i \neq j).$$

Duality

Unbiased $\Longrightarrow$ Orthogonal

If $|\psi_i\rangle$ and $|\psi_j\rangle$ are from different bases, then

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n}, \quad \vec{v}_i \cdot \vec{v}_j = 0 \quad (i \neq j).$$

Orthogonal $\Longrightarrow$ Unbiased

If $|\psi_i\rangle$ and $|\psi_j\rangle$ are from the same basis, then

$$|\langle\psi_i|\psi_j\rangle|^2 = 0, \quad \vec{v}_i \cdot \vec{v}_j = -\frac{1}{n-1} \quad (i \neq j).$$

SIC POVM

SIC POVM

Symmetric Informationally Complete POVM

A set of n^2 unit vectors $|\psi_i\rangle \in \mathbb{C}^n$, such that

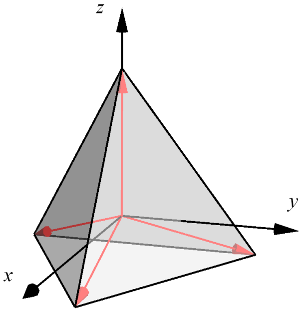
$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n+1}, \quad \vec{v}_i \cdot \vec{v}_j = -\frac{1}{n^2-1} \quad (i \neq j).$$

SIC POVM

Symmetric Informationally Complete POVM

A set of n^2 unit vectors $|\psi_i\rangle \in \mathbb{C}^n$, such that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n+1}, \quad \vec{v}_i \cdot \vec{v}_j = -\frac{1}{n^2-1} \quad (i \neq j).$$

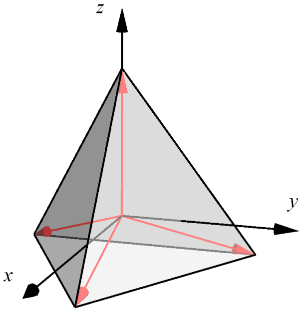


SIC POVM

Symmetric Informationally Complete POVM

A set of n^2 unit vectors $|\psi_i\rangle \in \mathbb{C}^n$, such that

$$|\langle\psi_i|\psi_j\rangle|^2 = \frac{1}{n+1}, \quad \vec{v}_i \cdot \vec{v}_j = -\frac{1}{n^2-1} \quad (i \neq j).$$



$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$|\psi_2\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix},$$

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{+i\varphi}\sqrt{2} \end{pmatrix},$$

$$|\psi_4\rangle = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ e^{-i\varphi}\sqrt{2} \end{pmatrix},$$

where $\varphi = \frac{2\pi}{3}$.

Thank you for your attention!